HW 8: ANALYSIS

1. The sequence $a_0, a_1, a_2, ...$ satisfies

$$a_{m+n} + a_{m-n} = \frac{a_{2m} + a_{2n}}{2}$$

for all nonnegative integers m and n with $m \geq n$. If $a_1 = 1$, determine a_n.

2. Compute

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + ...}}}}$$

3. Show that the series

$$\frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \frac{8}{1+x^8} + ... + \frac{2^n}{1+x^{2^n}} + ...$$

converges when $|x| > 1$, and in this case find its sum.

4. Let $a_n = \sqrt{1 + (1 + \frac{1}{n})^2} + \sqrt{1 + (1 - \frac{1}{n})^2}$, $n \geq 1$. Prove that

$$\frac{1}{a_1} + ... + \frac{1}{a_{20}}$$

is an integer.

5. Let a and b be real numbers in the interval $(0, \frac{1}{2})$ and let f be a continuous real-valued function such that

$$f(f(x)) = af(x) + bx,$$

for all $x \in \mathbb{R}$. Prove that $f(0) = 0$.

6. Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)dx = 0$. Prove that for every $\alpha \in (0, 1)$:

$$\int_0^\alpha f(x)dx \leq \frac{1}{8} \max_{x \in [0,1]} |f'(x)|.$$

7. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a)f'(a)f''(a)f'''(a) \geq 0.$$